S.-T. Yau College Student Mathematics Contests 2025

Oral Exams in Geometry and Topology

Individual (Solve 3 out of 4 problems)

1. Let S^n be the unit n-sphere and T^*S^n be its cotangent bundle.

(1) Show that T^*S^n is a complex submanifold of $\mathbb{C}^{n+1}$.

(2) Is T^*S^n, with its standard symplectic structure, symplectomorphic to a symplectic submanifold of $\mathbb{C}^{n+1}$? Show your answer explicitly.

2. Consider the 2-form

$$\Omega = \frac{dz_1 \wedge dz_2}{z_1 z_2}$$

on $(\mathbb{C}^\times)^2$, where $\mathbb{C}^\times = \mathbb{C} - \{0\}$ denotes the punctured complex plane. Ω is extended as a meromorphic 2-form on the complex projective plane $\mathbb{CP}^2$.

(1) What is the pole divisor of Ω on $\mathbb{CP}^2$? Show your assertion.

(2) Consider the disc map $u : \Delta \to \mathbb{CP}^2$, where $\Delta \subset \mathbb{C}$ is the unit disc, and

$$u(\zeta) = (z_1(\zeta), z_2(\zeta)) = \left(\frac{\zeta - a}{1 - \bar{a}\zeta}, \frac{\zeta - b}{1 - \bar{b}\zeta} \right)$$

where a, b are constants that satisfy $|a|, |b| < 1$, and (z_1, z_2) are the inhomogeneous coordinates of $\mathbb{CP}^2$. Show that the image of the disc boundary $\partial\Delta$ lies in the torus defined by $|z_1| = |z_2| = 1$.

(3) Could you make sense of the 'pullback' $u^*\Omega$ (rather than just being simply zero), and how many poles does $u^*\Omega$ have?

3. Let $\Sigma \subset \mathbb{R}^n$ be a complete minimal surface in $\mathbb{R}^n$.

(1) Show that the coordinate function $x_i (i = 1, \ldots, n)$ is harmonic on Σ.

(2) Prove that Σ is not closed.

4. The Einstein tensor on a Riemannian manifold with dimension≥ 2 is defined as

$$G = \mathrm{Ric} - \frac{\mathrm{scal}}{2} g.$$

(1) Show that div$G$ =0.

(2) Show that $G = 0$ in dimension 2 and that $G = 0$ in dimensions> 2 if and only if the metric is Ricci-flat.